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MINIMAL CLASSES AND MAXIMAL CLASS IN p-GROUPS

BY

PATRIZIA LONGOBARDI

Dipartimento di Matematica e Applicazioni, Università degli Studi di Napoli
Monte S. Angelo, via Cintia, 80126 Napoli, Italy
e-mail: maj@matna2.dma.unina.it

AND

MERCEDE MAJ

Dipartimento di Ingegneria dell'Informazione e Matematica Applicata
Università degli Studi di Salerno, via S. Allende, 84081 Baronissi (Salerno), Italy
e-mail: maj@matna2.dma.unina.it

AND

AVINOAM MANN*

Einstein Institute of Mathematics, The Hebrew University of Jerusalem
Givat Ram, Jerusalem 91904, Israel
e-mail: mann@math.huji.ac.il

ABSTRACT

The number of conjugacy classes of a given size (not 1) in a p-group is divisible by $p-1$. We study groups in which the number of classes of minimal size is exactly $p-1$, and characterise metabelian groups and groups of maximal class with this property.

In this paper all groups are non-abelian finite p-groups. We start with some simple observations on the sizes and number of conjugacy classes in such groups. Recall that if G is a finite p-group, and p^b is the size of the conjugacy class of $x \in G$, we term b the **breadth** of x, denoted $b(x)$. The **breadth** $b(G)$ of G

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is the maximal breadth of its elements. It is easy to see that the number of (non-identity) classes of a given breadth is divisible by $p-1$. It is then a natural task to investigate the groups in which this number is exactly $p-1$. In case the given breadth is $b(G)$, this question was addressed in [Mc2], where it is shown that if G, of order p^n, has $p-1$ classes of size $b = b(G)$, then $\mathrm{cl}(G) \geq 3, b \geq 4$, and $n \leq b^2 + b$. We remark that the last inequality can be improved slightly, to $n \leq b^2$, by choosing, in the notation of the proof of Theorem 1 of [Mc2], the elements $b_1, ..., b_n$ outside a maximal subgroup containing $C(a)$, hence of non-maximal breadth. Macdonald also gives some examples of such groups, and further examples are provided by the groups in [VW]. It is worth noting that in all these examples we have $p = 2$.

In the present paper our main interest is in the non-central classes of minimal breadth. We refer to such classes as **minimal classes**. We first show that there are always some minimal classes which are contained in $Z_2(G)$. Then we consider the groups which contain just $p-1$ minimal classes, and characterise metabelian groups with this property. These turn out to be closely related to groups of maximal class, hence our title. More precisely, these groups are **CF-groups**, i.e. groups G of class at least three in which all lower central factors G_i/G_{i+1} except the first one have order p, and obey some further restrictions (see Theorem 12). On the way we obtain characterisations of CF metabelian groups, and of metabelian groups of maximal class (see Corollary 8 and Proposition 10).

Notation: Let $|G| = p^n$, let $1 < p^{s_1} < \cdots$ be the sizes of the conjugacy classes of G, and write $s = s_1$. Write $Z = Z(G), |Z| = p^z$, and assume that G contains u_i classes of size p^{s_i}. Write $u = u_1$.

1. Conjugacy classes

PROPOSITION 1:
(1) *For each i we have $p-1 \mid u_i$ and $p^z \mid u_i p^{s_i}$.*
(2) *Either $p^s = p^z$, and $u \equiv p-1 \pmod{p(p-1)}$, or $p^s < p^z$, and $u \equiv 0 \pmod{p(p-1)}$.*

Proof: (1) Let p^e be the exponent of G. If $(p,j) = 1$, then the map $x \to x^j$ induces a permutation of the classes of G, and thus the group of residues prime to p^e, of order $p^{e-1}(p-1)$, acts as a permutation group on the set of classes of G. If the class of x is invariant under the above permutation, then x is conjugate to x^j, which is possible only if $j \equiv 1 \pmod p$, and then the order of j in the group of residues is a power of p. It follows that the orbit of each class has size divisible by $p-1$, and the same holds for the set of classes of a fixed size, which is a union of orbits.

The other divisibility claim follows from the fact that the set of elements of breadth p^{s_i} has cardinality $u_i p^{s_i}$ and is a union of cosets of $Z(G)$.

(2) The class equation of G is

$$p^n = p^z + up^s + u_2 p^{s_2} + \cdots,$$

which implies that $p^s \mid p^z$. Dividing by p^s, we see that $p \mid p^{z-s} + u$, yielding our claim. ∎

Passing now to minimal classes, we first note the following obvious facts: *p^s is the minimal index of proper centralisers in G, in other words, the minimal index of subgroups H such that $Z(H)$ is not contained in $Z(G)$. The minimal classes consist exactly of the elements in the centres of such subgroups which are not in $Z(G)$.*

This characterisation of the minimal size is analogous to the following characterisation of the minimal degree of non-linear irreducible characters (see [M3]): *the minimal degree is the minimal index of a subgroup H such that $H' \neq G'$, and the minimal characters are the characters of G that are induced from linear characters of such subgroups H which cannot be extended to linear characters of G.* In [M3] it is also shown that the relevant subgroups of minimal index are normal. The corresponding claim, that the centralisers of minimal index are normal, is not true. Thus, let p be odd, and let $G = \langle x, y \rangle$ with relations $x^p = y^p = 1$ and all commutators of weight 4 trivial. Then $|G| = p^5, |Z(G)| = |G_3| = p^2$ and $|Z_2(G)| = |G'| = p^3$. All non-central elements have centralisers of order p^3, but the only normal centraliser is $Z_2(G)$. Note the following:

A maximal centraliser C is normal, if and only if it is the centraliser of a non-central element of $Z_2(G)$.

Indeed, if $x \in Z_2(G)$, then $C(x) \geq G'$, so $C(x) \lhd G$. Conversely, if $C \lhd G$, then $Z(C) \lhd G$, and $Z(C)$ contains $Z(G)$ properly, therefore there exists an element $x \in Z(C) \cap (Z_2(G) \setminus Z(G))$, and the maximality implies $C = C(x)$.

Note that this does not mean that if $C(x) \lhd G$, then $x \in Z_2(G)$. Some properties of elements with normal centralisers are discussed in [M2].

THEOREM 2: *Let G be a non-abelian p-group. Then some of the minimal classes of G are contained in $Z_2(G)$; equivalently, some centralisers of maximal order are normal. Moreover, if x is an element of minimal breadth, there exists an element w of minimal breadth in $Z_2(G)$ of the form $[x, y, \ldots, y]$, for some y.*

Proof: Let x be an element of a minimal class, and write $C = C(x)$, so $|G : C| = p^s$. If $p^s = p$, then $C \triangleleft G$, so we assume that $s > 1$. Let $y \in N(C) - C$ be such that $y^p \in C$. Writing $[x, y; n]$ for $[x, y, y, ..., y]$, where y occurs n times, let k be minimal such that $[x, y; k] = 1$. Then $k > 1$. Now $x \in Z(C) \triangleleft N(C)$, so $[x, y; k-1] \in Z(C)$, and $C([x, y; k-1])$ contains both C and y. The maximality of C implies that $[x, y; k-1] \in Z(G)$. Write $z = [x, y; k-2]$. Then $[z, y] \neq 1$, so $z \notin Z(G)$. But $z \in Z(C)$, so z belongs to a minimal class. Moreover $[z, y] \in Z(G)$, therefore $\langle z, y \rangle$ is of class 2, and $y^p \in C = C(z)$, implying $[z, y]^p = 1$. Writing $N = \langle [z, y] \rangle$ and $H = G/N$, it follows that in H the conjugacy class of zN has size p^{s-1}, and that this class is a minimal one in H. By induction, there exists an element w such that $wN \in Z_2(H)$, and wN lies in a minimal class of H. Then the class of wN has size p^{s-1}, so the class of w in G has size at most p^s, and by minimality that size is exactly p^s. Let $v \in G$. Then $[w, v]N \in Z(H)$, so the class of $[w, v]$ in G is contained in $[w, v]N$, and has size at most p. Since $p^s > p$, we have $[w, v] \in Z(G)$ and $w \in Z_2(G)$.

To prove the statement about the shape of w, let again first $s = 1$, then $Z(C) \triangleleft G$, therefore all elements of the form $[x, y, ..., y]$ are in $Z(C)$, where we choose again $y \notin C$. Suppose that $x \in Z_i(G) \setminus Z_{i-1}(G)$, for some $i > 2$. Then, since x is not central (mod $Z_{i-2}(G)$), it follows that $[x, y] \in Z_{i-1}(G) - Z_{i-2}(G)$, and $[x, y]$ is also of breadth 1. Continuing in the same way, we see that $[x, y; n] \in Z_2(G) - Z(G)$, for some n. For $s > 1$, w has the required form by induction. ∎

This theorem enables us to give another characterisation of the minimal breadth, which is dual to the characterisation of minimal degree.

PROPOSITION 3:

(1) *Any normal subgroup of G of order at most p^s is central.*

(2) *p^s is the minimal order of a normal subgroup N such that $Z(G/N) \neq Z(G)/N$.*

Proof: Let $N \triangleleft G$ and $|N| \leq p^s$. Then the elements of N have less than p^s conjugates, so they are central. Moreover, if $x \in Z(G \bmod N)$, then all conjugates of x are contained in xN, so the number of conjugates is at most $|N|$. Thus, if $|N| < p^s$, then x is central. On the other hand, let z be an element of $Z_2(G)$ which belongs to a minimal class. Then $N := [x, G]$ is a normal subgroup of order p^s and $x \in Z(G \bmod N)$, so $Z(G \bmod N) \neq Z(G)$. ∎

PROPOSITION 4:

(1) *If $x \in Z_2(G)$ has minimal breadth, then $x^p \in Z(G)$.*

(2) *Let N_1 and N_2 be two normal subgroups of order p^s such that $Z_i :=$ $Z(G \bmod N_i) \neq Z(G)$. Then $Z_1 \cap Z_2 = Z(G)$.*
(3) *Similarly, if C_1 and C_2 are two maximal centralisers, and $Z_i = Z(C_i)$, then $Z_1 \cap Z_2 = Z(G)$.* ∎

Proof: (1) Let x be an element of $Z_2(G)$ that lies in a minimal class, let $C = C(x)$, and let $y \notin C$, but $y^p \in C$. Then $\langle x, y\rangle$ is of class 2, therefore $[x^p, y] = [x, y^p] = 1$. Thus $C(x^p)$ contains $C(x)$ properly, so x^p is central.

(2) Let $x \in Z_1 \cap Z_2$. Then $[x, G] \leq N_1 \cap N_2$, so x has less than p^s conjugates, and is central.

A similar proof establishes (3). ∎

PROPOSITION 5: *Let G be a group of class c having exactly $p-1$ minimal classes. Then:*

(1) *$Z_2(G)$ is elementary abelian, $|Z_2(G) : Z(G)| = p$, and the minimal classes are the classes lying in $Z_2(G) - Z(G)$. These classes are cosets of $Z(G)$.*
(2) *If $x \in Z_3(G) - Z_2(G)$, then the class of x is $xZ_2(G)$.*
(3) *$c \geq 4$. Moreover, $G_{c-1} = Z_2(G)$ and $G_c = Z(G)$.*

Proof: It is clear from Proposition 1 that $u = p - 1$ is possible only if $p^s = p^z$. Now if $x \in Z_2(G) - Z(G)$, then the conjugacy class of x is contained in $xZ(G)$, so its size is at most p^z. Thus this class must have size p^z, and it coincides with the coset $xZ(G)$. Since different cosets of $Z(G)$ in $Z_2(G)$ yield different classes, there are at most $p - 1$ such cosets, and $|Z_2(G) : Z(G)| = p$. Moreover, these cosets are all the minimal classes of G. Therefore if $x \in Z_3(G) - Z_2(G)$, then the class of x has size at least p^{s+1}. But this class is contained in $xZ_2(G)$, of size p^{z+1}, so the class of x is $xZ_2(G)$.

Letting again $x \in Z_2(G) - Z(G)$, and $z \in Z(G)$, we have $z = [x, u]$, for some u, hence $z^p = [x^p, u] = 1$. Thus $Z(G)$ is elementary abelian, while $Z_2(G) = \langle x, Z(G)\rangle$ is abelian. Suppose that $Z_2(G)$ is not elementary. Then there exists a maximal subgroup W of $Z(G)$ such that $Z_2(G)/W$ is cyclic of order p^2. The fact that elements of $Z_2(G)$ are conjugate to all elements in their coset of $Z(G)$ shows that $Z(G/W) = Z(G)/W$. Thus $Z_2(G/W) = Z_2(G)/W$ is cyclic, which implies that G/W is dihedral, quaternion, or semidihedral [Hu, III.7.7]. Since the classes of G/W have size at least the size of the corresponding class in G, divided by p^{s-1}, we see that G/W also has just $p - 1$ minimal classes (of size p). This is incompatible with the structure of G/W. Thus $Z_2(G)$ is elementary.

If $G = Z_2(G)$, then G is abelian. Suppose $\mathrm{cl}(G) = 3$. For each $x \in Z_3(G) - Z_2(G)$, the elements of $Z_2(G)$ are commutators of x. Thus $G' = Z_2(G)$. Then the fact that the class of x is $xZ_2(G)$ means that G is a so-called Camina group, and such groups cannot have $|Z_2(G) : Z(G)| = p$ [Mc1]. Therefore $\mathrm{cl}(G) \geq 4$.

There exists an element $x \in G_{c-2} \leq Z_3(G)$, such that $x \notin Z_2(G)$. Then (2) shows that $Z_2(G) = [x, G] \leq G_{c-1}$. Similarly, changing x to a non-central element in $Z_2(G)$ shows that $Z(G) \leq G_c$. ∎

2. Metabelian groups

In this section we derive some properties of metabelian p-groups, which will then be used for characterising such groups with just $p-1$ minimal classes. We write Z_i and Z for $Z_i(G)$ and $Z(G)$.

PROPOSITION 6: *Let G be a metabelian group of class at least 4 in which $|Z_2(G) : Z(G)| = p$. Then the elements of $Z_2(G) \setminus Z(G)$ have breadth 1.*

Proof: Let G have class c. Since $c \geq 4$, we have $G_{c-2} \leq Z_3 \cap G'$, but $G_{c-2} \not\leq Z_2$, so we can choose an element $x \in Z_3 \cap G' - Z_2$, and there exists an element y such that $[x, y] \notin Z$. Then $[x, y]$ generates $Z_2(G) \pmod{Z(G)}$, and all elements of $Z_2(G) - Z(G)$ are of the form $[x, y]^i z, 0 < i < p, z \in Z(G)$, so all these elements have the same centraliser. Let $C = C(x \bmod Z(G))$; then $|G : C| = p$, and our claim will be proved by showing that $C([x, y]) = C$. This is an immediate corollary of the three subgroups lemma, according to which $[x, G, C] \leq [C, x, G][G, C, x] = 1$. ∎

PROPOSITION 7: *Let G be a metabelian group of class $c > 2$ such that $|Z_2(G) \cap G'| = p^2$. Then $Z_i(G) \cap G' = G_{c+1-i}$, for $1 \leq i \leq c-1$, and G is a CF-group.*

Proof: The claims are obvious for $i = 1, 2$, and we proceed by induction. Assuming the claims valid for i and all smaller indices, we divide by $Z_{i-2}(G) \cap G'$, and replace G by the factor group. That means that we have only to show our claims for $i = 3$.

Let $x \in Z_3 \cap G' - Z_2$, and let y be an element such that $[x, y] \notin Z$. The proof of the previous proposition shows that $C := C(x \bmod Z) = C([x, y]) = C(Z_2 \cap G')$, and that this subgroup has index p. Therefore $G = \langle C, y \rangle$. If also $u \in Z_3 \cap G' - Z_2$, then similarly $C(u \bmod Z) = C$. Also, $[u, y] = [x, y]^e z$, for some integer e and some central element z. Thus $[ux^{-e}, y] \in Z$ and $[ux^{-e}, G] = [ux^{-e}, C\langle y \rangle] \leq Z$, so $ux^{-e} \in Z_2$. This shows that $Z_3 \cap G'/Z_2 \cap G'$ is cyclic, generated by $x(Z_2 \cap G')$. But $[x, y]^p \in Z$, implying $[x^p, y] \in Z$ and $x^p \in Z_2$, so that $|Z_3 \cap G' : Z_2 \cap G'| = p$.

Since $G_{c+1-i} \leq Z_i \cap G'$ and $|G_{c+1-i}| \geq p^i \geq |Z_i \cap G'|$, the two subgroups are equal, and the calculation of the indices that was just made shows that G is a

CF-group. ∎

COROLLARY 8: *A metabelian group G of class $c > 2$ is a CF-group if and only if $|Z_2(G) \cap G'| = p^2$.*

COROLLARY 9: *Let G be a metabelian group of class $c > 2$ in which $|Z_2 \cap D| = p^2$, where $D = C(G')$. Then G is a CF-group, and $Z_i \cap D = Z_i \cap G'$, for $1 \leq i \leq c-1$.*

Proof: G is CF by the previous proposition, and an argument similar to the proof of that proposition shows that $|Z_i \cap D : Z_{i-1} \cap D| = p$. ∎

It is of interest to note also the following result, a sort of dual to the last one, though it is not needed for the rest of the paper.

PROPOSITION 10: *Let G be a two generator metabelian group of class c, and suppose that $|G_i : G_{i+2}| = p^2$, for some $i \leq c-1$. Then $|G_k : G_{k+1}| = p$, for all $i \leq k \leq c$. In particular, if G/G_4 is of maximal class, so is G.*

This is proved (though not stated) by the proof of Proposition 7.d in [Br].

3. Minimal classes

In this section we need more terminology. Let G be a CF-group. We write $G_1 = C(G_2/G_4)$. Then $|G : G_1| = p$. If G_1 is non-abelian, we say that G has **degree of commutativity** k, if $[G_i, G_j] \leq G_{i+j+k}$, for all indices such that $i + j + k \leq c + 1$, but for at least one such pair i, j we have $[G_i, G_j] \not\leq G_{i+j+k+1}$. (If G_1 is abelian, we say that G has degree of commutativity $c-2$.)

LEMMA 11: *Let G be a metabelian CF-group of class $c > 3$. If $[G_1, G_i] \leq G_{i+1+k}$, for some $2 \leq i \leq c-2, k \leq c-i$, then $[G_1, G_j] \leq G_{j+1+k}$ for all $j \geq 1$.*

Proof: First,

$$[G_{i+1}, G_1] = [G_i, G, G_1] \leq [G_1, G_i, G][G, G_1, G_i] = [G_1, G_i, G] \leq G_{i+2+k},$$

so induction yields the claim for $j \geq i$. Next,

$$[G_1, G_{i-1}, G] \leq [G, G_1, G_{i-1}][G_{i-1}, G, G_1].$$

If $i-1 \geq 2$, then $[G, G_1, G_{i-1}] = 1$, while if $i-1 = 1$, then $[G, G_1, G_{i-1}] = [G_2, G_1]$. In either case we have $[G_1, G_{i-1}, G] \leq [G_i, G_1] \leq G_{i+1+k} \leq Z_{c-i-k}$, so $[G_1, G_{i-1}] \leq G' \cap Z_{c+1-i-k} = G_{i+k}$, and reverse induction yields the claim also for $j \leq i$. ∎

THEOREM 12: *A metabelian group G of class c has just $p-1$ minimal classes if and only if G is a CF-group of degree of commutativity 1 satisfying $c \geq 4$ and $C(G') \cap Z_{c-1} = G'$.*

Proof: First let G have $p-1$ minimal classes. Combining Propositions 5 and 6 shows that $c \geq 4$ and that $|Z_2(G)| = p^2$, and then Corollary 9 shows that G is a CF-group and that $C(G') \cap Z_{c-1} = G'$. By the corollary to [Bl, Theorem 2.10], G has positive degree of commutativity, say k. If $k > 1$, then $[G_{c-2}, G_1] \leq G_{c-2+1+k} = 1$, so all elements of G_{c-2} have at most p conjugates, yielding at least $p^2 - 1$ minimal classes. Thus $k = 1$.

Now let G be a CF-group of class at least 4, degree of commutativity 1, and satisfying $C(G') \cap Z_{c-1} = G'$. We wish to show that all elements of breadth 1 lie in $Z_2(G)$. Let $b(x) = 1$, and at first assume also that $x \in G'$. Then $x \in G_i \setminus G_{i+1}$, for some i, and then $G_i = \langle x \rangle G_{i+1}$. Assume that $i \leq c-2$. By the lemma, $[G_1, G_i] \not\leq G_{i+3}$, so by assumption $[G_1, G_i] = G_{i+2}$. Thus G_1 does not centralise x, so since $b(x) = 1$, we have $[G_1, x] = [G, x]$. Then $G_{i+1} = [G, G_i] = [G, \langle x \rangle G_{i+1}] \leq [G, x][G, G_{i+1}] = [G_1, x]G_{i+2} = G_{i+2}$, a contradiction. Thus $x \in G_{c-1}$, as needed. Now suppose that $x \notin G'$. Because x has breadth 1, it centralises G', which implies that $[x, y, z] = [x, z, y]$ for any two elements y, z, and in particular $C(x) \leq C([x, y])$. Then the commutators $[x, y]$ also have breadth 1, and lie in Z_2, by what was proved already. Then $x \in Z_3 \cap C(G')$, so $x \in G'$, and we are done. ∎

Recall that an **ECF-group** is a CF-group in which G/G' is elementary.

COROLLARY 13: *A metabelian ECF-group G with just $p-1$ minimal classes is of class $p+1$ at most. In particular, if G is of maximal class, its order is at most p^{p+2}.*

Proof: By [Bl, 3.10], G has degree of commutatitivity at least $c - p$.

We now discuss groups of maximal class that are not necessarily metabelian. These will provide us with several examples of groups with just $p-1$ minimal classes, in particular they show that there are such groups for all orders up to p^{p+2}. Most of these examples are not metabelian.

If G is of maximal class and order p^n, then $G_1 = C(G_i/G_{i+2})$ for $2 \leq i \leq n-3$ [Hu, III.14.6]. If also $G_1 = C(G_{n-2})$, then G has positive degree of commutativity, and is termed **non-exceptional**, otherwise it is **exceptional**. Exceptional groups exist if and only if n is even and satisfies $5 \leq n \leq p+1$ [Hu, III.14.6, III.14.24].

Next, let G be any group of order p^n, and write $n = 2m + e$, $e = 0, 1$. Then a formula of P. Hall states that the number of conjugacy classes of G is $1 + e(p-1) + m(p^2 - 1) + r(p-1)(p^2 - 1)$, for some non-negative integer r, called the **abundance** of G. [Hu, V.15.2, or M1]. If $r = 0$, then G is of maximal class, $n \leq p + 1$ if $p \geq 11$, and $n \leq p + 2$ always ([Po] and [FS]). Such groups exist for all orders p^n in the above ranges, according to the constructions in [Pa, KLG, Yo, Ro].

PROPOSITION 14: *Let G be of maximal class and order p^n. Then G contains exactly $p - 1$ minimal classes if and only if $[G_1, G_{n-3}] \neq 1$. This includes all exceptional groups, and also all groups of abundance 0, and implies $n \leq 2p - 3$.*

Proof: Let $b(x) = 1$, and let $C = C(x)$. Then $x \in Z(C) \triangleleft G$. Thus $Z(C)$ is a normal subgroup of order at least p^2, and therefore contains G_{n-2}, and $C = C(G_{n-2})$. It follows that G has $p - 1$ minimal classes if and only if $Z(C) = G_{n-2}$. If this is not the case, then $Z(C) \geq G_{n-3}$. This cannot happen if G is exceptional, because then C does not centralise even G_{n-3}/G_{n-1}. If G is non-exceptional, then $C = G_1$, so $Z(C) = G_{n-2}$ is equivalent to $[G_1, G_{n-3}] \neq 1$, and the latter inequality holds also for exceptional groups. Moreover, if G has positive degree of commutativity k, that inequality shows that $k = 1$, and [Fe] shows that $n \leq 2p - 3$.

Groups of abundance 0 are characterised in [VF, (2.1)], which shows that these groups are non-exceptional and satisfy $[G_1, G_{n-3}] = G_{n-1}$. This concludes the proof. ∎

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